Anomalous Hall Effect and Skyrmion Number in Real- and Momentum-space

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We study the anomalous Hall effect (AHE) for the double exchange model with the exchange coupling $|J_H|$ being smaller than the bandwidth $|t|$ for the purpose of clarifying the following unresolved and confusing issues: (i) the effect of the underlying lattice structure, (ii) the relation between AHE and the skyrmion number, (iii) the duality between real and momentum spaces, and (iv) the role of the disorder scatterings; which is more essential, σ_H (Hall conductivity) or ρ_H (Hall resistivity)? Starting from a generic expression for σ_H, we resolve all these issues and classify the regimes in the parameter space of $J_H\tau$ (τ: elastic-scattering time), and λ_s (length scale of spin texture). There are two distinct mechanisms of AHE; one is characterized by the real-space skyrmion-number, and the other by momentum-space skyrmion-density at the Fermi level, which work in different regimes of the parameter space.

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The anomalous Hall effect (AHE) is a phenomenon where the Hall resistivity has an additional contribution due to the spontaneous magnetization in ferromagnets. This anomalous contribution has been attributed to the spin-orbit interaction, and various mechanisms has been proposed [1, 2, 3, 4]. Recently it has been recognized that the original expression by Karplus and Luttinger [1], i.e., the intrinsic contribution, has the geometrical meaning in terms of the Berry-phase curvature in momentum space [5, 6, 7]. This is analogous to the the integer quantum Hall effect (IQHE) with the strong external magnetic field [8, 9]. It was also proposed that AHE arises even without the spin-orbit interaction if the spin configuration is non-coplanar with finite spin chirality, i.e., the solid angle subtended by the spins where the electron hops successively [10, 11, 12, 13, 14]. Consider the double-exchange model

$$H = \sum_{\langle \boldsymbol{r},\boldsymbol{r}'\rangle} t\, \boldsymbol{c}^\dagger_{\boldsymbol{r}}\boldsymbol{c}_{\boldsymbol{r}'} - \frac{J_H}{2}\sum_{\boldsymbol{r}} \boldsymbol{S}_{\boldsymbol{r}}\cdot\left[\boldsymbol{c}^\dagger_{\boldsymbol{r}}\boldsymbol{\sigma}\boldsymbol{c}_{\boldsymbol{r}}\right] \qquad (1)$$

where $\langle \boldsymbol{r},\boldsymbol{r}'\rangle$ runs the nearest neighbor sites, $\boldsymbol{c}^{(\dagger)}_{\boldsymbol{r}} = (c^{(\dagger)}_{\boldsymbol{r}\uparrow}, c^{(\dagger)}_{\boldsymbol{r}\downarrow})$ is the annihilation (creation) operator at the site $\boldsymbol{r}$, and $\boldsymbol{S}_{\boldsymbol{r}}$ is the classical spin localized at the site $\boldsymbol{r}$. Assuming a strong Hund coupling $|J_H|(\gg |t|)$ between the conduction electrons and the localized spins, the Berry phase of the localized spins acts as a fictitious magnetic field for the conduction electron [15, 16, 17]. Ye *et al.* assumed that this fictitious magnetic field has a uniform component due to the spin-orbit interaction in the presence of the uniform magnetization [10]. However there is a subtle issue concerning the definition of the skyrmion number when the spins are defined on the discrete points and/or the underlying lattice is relevant. This is related to the length scale with respect to the spin texture and/or the lattice structure. Furthermore, the effect of the spin-orbit interaction can not be represented by the spatially uniform magnetic field; it induces the effective "magnetic field", i.e., the Berry phase curvature, in *momentum* space. In real systems, the disorder is also relevant and often the following question arises: Which is more essential, the Hall conductivity σ_H or the Hall resistivity ρ_H? Therefore it is highly desirable to resolve all these issues in a unified fashion by clearly articulating the connection between the AHE and the skyrmion number. In this paper, we study the AHE for the double exchange model eq. (1) in the case of the small exchange coupling $|J_H|$ compared with the bandwidth $|t|$.

It is found that the non-coplanar spin-configuration induces the AHE through the two distinct mechanisms originated by ($\mathrm{M_I}$) the non-zero topological-windings of spin texture, and ($\mathrm{M_{II}}$) the nontrivial structure of underlying lattice. Here the nontrivial structure means that the Wigner-Seitz unit cell contains multiple sites and different kinds of plaquettes. Although the two mechanisms work simultaneously in generic cases, we can identify the two distinct limits where these mechanisms works exclusively:
$\mathrm{C_I}$: The unit cell contains a single site and the length scale λ_s of the spin texture is sufficiently longer than the underlying lattice constant a ($\lambda_s \gg a$).
$\mathrm{C_{II}}$: The unit cell contains multiple sites and the periodicity of the spin texture is the same as the underlying lattice ($\lambda_s = a$).

The mechanism $\mathrm{M_I}$ is dominant and the AHE is characterized by the real-space skyrmion-number in the case $\mathrm{C_I}$, while $\mathrm{M_{II}}$ is dominant and is characterized by the momentum-space skyrmion-density at the Fermi level in $\mathrm{C_{II}}$. In the latter case, the AHE takes place even if the spin texture has no winding in real space and the total skyrmion-number is zero in momentum space.

We also discuss the effect of disorder which causes the

following crossover. The Hall conductivity σ_H is proportional to $|J_H^3|\tau^2/|t|$ for $|J_H| \ll |t|(a/\lambda_s)^d \ll 1/\tau \ll |t|(a/\lambda_s)$ in the case $\mathrm{C_I}$, and for $|J_H| \ll 1/\tau$ in the case $\mathrm{C_{II}}$. Here τ is the elastic-scattering time, and d is the dimensionality of a system. Hence the Hall resistivity, $\rho_H = -\sigma_H/(\sigma_0^2+\sigma_H^2)$ ($\sigma_0 \propto \tau$, $\sigma_0 \gg |\sigma_H|$), does not depend on $\rho_0 \cong 1/\sigma_0$, where σ_0 and ρ_0 are diagonal conductivity and resistivity respectively. However, when $1/\tau \ll |J_H|$, σ_H approaches its intrinsic value, and ρ_H is proportional to ρ_0^2 even in the case where $|J_H|$ is sufficiently smaller than the band width. This suggests that the intrinsic (i.e., not due to impurity scattering) meaning of the AHE can be observed through ρ_H [18] even in the weak coupling regime.

In order to consider the present issues with taking into account the effect of disorder, it is transparent to start with the Středa formula [19],

$$\begin{aligned}\sigma_{\mu\nu} &= -\frac{1}{2\pi V}\int dE f_F(E) \\ &\quad\times\left\langle \mathrm{Tr}\left[J_\mu \frac{dG_+}{dE}(E) J_\nu G_\delta(E) + \mathrm{H.c.}\right]\right\rangle_{\mathrm{imp}}, \quad (2)\end{aligned}$$

where V the system volume, $\boldsymbol{J}$ the current operator, $G_\pm(E) = [E - H - H_{\mathrm{imp}} \pm i\eta]^{-1}$, $G_\delta = G_+ - G_-$, H_{imp} being the interaction with impurities, and $\langle\cdots\rangle_{\mathrm{imp}}$ represents the ensemble average over impurity configuration. Here and hereafter we take the unit where $c = \hbar = 1$.

In this paper, we shall focus on only metallic states ($|t|\tau \gg 1$), and just replace $G_\pm$ by $[E - H \pm i(2\tau)^{-1}]^{-1}$ to bring in the effect of disorder, i.e. assuming the isotropic impurity potential and neglecting the localization effect. At zero temperature and using the symmetry $\sigma_{\nu\mu}|_{\mu\neq\nu} = -\sigma_{\mu\nu}|_{\mu\neq\nu}$, the Hall conductivity is given by

$$\begin{aligned}\sigma_H &\cong -\frac{i}{V}\sum_{\alpha,\alpha'} \boldsymbol{e}_\perp \cdot [\langle\boldsymbol{\psi}_\alpha|\boldsymbol{J}|\boldsymbol{\psi}_{\alpha'}\rangle \times \langle\boldsymbol{\psi}_{\alpha'}|\boldsymbol{J}|\boldsymbol{\psi}_\alpha\rangle] \\ &\quad\times\frac{1}{\pi}\left[\frac{\arctan(-2\tau\xi_\alpha) - \arctan(-2\tau\xi_{\alpha'})}{(\xi_\alpha - \xi_{\alpha'})^2}\right. \\ &\quad\left.+\frac{2\tau(1+4\tau^2\xi_\alpha\xi_{\alpha'})}{(\xi_\alpha-\xi_{\alpha'})[1+(2\tau\xi_\alpha)^2][1+(2\tau\xi_{\alpha'})^2]}\right] \\ &\cong -\frac{i}{V}\sum_{\alpha,\alpha'} \boldsymbol{e}_\perp \cdot [\langle\boldsymbol{\psi}_\alpha|\boldsymbol{J}|\boldsymbol{\psi}_{\alpha'}\rangle \times \langle\boldsymbol{\psi}_{\alpha'}|\boldsymbol{J}|\boldsymbol{\psi}_\alpha\rangle] \\ &\quad\times\frac{\tau^2\left[f_F^\tau(E_\alpha) - f_F^\tau(E_{\alpha'})\right]}{1+\left[(E_\alpha - E_{\alpha'})\tau\right]^2}, \quad (3)\end{aligned}$$

where $\xi_\alpha = E_\alpha - \mu_0$ (μ_0 : the chemical potential), $f_F^\tau(E)$ is $f_F(E)$ with replacing the inverse temperature by $64\tau/(3\pi)$. The unit vector $\boldsymbol{e}_\perp$ is normal to the plane determined by the Hall measurement. It is noted that σ_H depends on $\boldsymbol{e}_\perp$. The second formula in eq. (3) is an approximation for the first one by using physical implication, and it is numerically confirmed that this approximation is almost exact with the accuracy less than 5% around peaks. In the clean limit, $\tau \to \infty$, eq. (3) reduces to the formula obtained in Refs. [8, 9].

Now we shall consider the AHE in the weak-coupling regime by assuming a periodic configuration of localized spins. Every lattice-site is classified into sublattices $I = A, B, C\cdots$. The localized spins satisfy $\boldsymbol{S}_{\boldsymbol{r}\in I} = \boldsymbol{S}_I$. Then the coupling with the localized spins, H', is expressed as

$$H' = -\frac{J_H}{2}\sum_{I=A,B,C,\cdots}\sum_{\boldsymbol{k}}^{1\mathrm{stBZ}} \boldsymbol{S}_I \cdot \left[\boldsymbol{c}_{I\boldsymbol{k}}^\dagger \boldsymbol{\sigma} \boldsymbol{c}_{I\boldsymbol{k}}\right], \quad (4)$$

where $\boldsymbol{c}_{I\boldsymbol{k}}^{(\dagger)}$ is a Fourier transformation of the annihilation (creation) operator at I-sublattice, $\boldsymbol{c}_{\boldsymbol{r}\in I}^{(\dagger)}$. The system has multiple bands even if we start with a single band at $J_H = 0$. The band separation is of the order of $|t|(a/\lambda_s)/(k_F\lambda_s)^{d-1}$ except for the spin splitting of the order of $|J_H|$, where k_F is the Fermi momentum in the case $J_H = 0$ and d is the dimensionality of the system. For simplicity, we shall consider the case $k_F \sim 1/a$ hereafter. When the parameter J_H satisfies $|J_H| \ll |t|(a/\lambda_s)^d$ and $\boldsymbol{k}$ is not near level crossings, approximated eigenvalues and eigenfunctions are obtained by the conventional perturbation theory for spin-degenerate systems,

$$\begin{aligned}E_{n\boldsymbol{k}\pm} &= E_{n\boldsymbol{k}} \mp \frac{J_H}{2}|\boldsymbol{S}_{n\boldsymbol{k}}| + \cdots, & (5)\\ \boldsymbol{S}_{n\boldsymbol{k}} &= \sum_{I=A,B,C,\cdots} \boldsymbol{S}_I |u_{n\boldsymbol{k},I}|^2, & (6)\\ |\boldsymbol{\psi}_{n\boldsymbol{k}\pm}\rangle &= \sum_{n'}\sum_{s=\pm}\sum_{l=0} c^{(l)}_{[n'\boldsymbol{k}s][n\boldsymbol{k}\pm]} |\boldsymbol{\psi}^{(0)}_{n'\boldsymbol{k}s}\rangle + \cdots, & (7)\\ |\boldsymbol{\psi}^{(0)}_{n\boldsymbol{k}\pm}\rangle &= \sum_{I=A,B,C,\cdots}\sum_{\sigma=\uparrow,\downarrow} u_{n\boldsymbol{k},I}\chi_{n\boldsymbol{k}\pm,\sigma} c^\dagger_{I\boldsymbol{k}\sigma}|0\rangle, & (8)\end{aligned}$$

where $c^{(l)}_{[n'\boldsymbol{k}s'][n\boldsymbol{k}s]}$ is the perturbative coefficient of the l-th order, $\boldsymbol{u}_{n\boldsymbol{k}} = (u_{n\boldsymbol{k},A}, u_{n\boldsymbol{k},B}, u_{n\boldsymbol{k},C}, \cdots)^T$ is the orbital part of an eigenfunction in $J_H = 0$, and $\boldsymbol{\chi}_{n\boldsymbol{k}\pm} = (\chi_{n\boldsymbol{k}\pm,\uparrow}, \chi_{n\boldsymbol{k}\pm,\downarrow})^T$ is the spin part of an eigenfunction satisfying $[\boldsymbol{S}_{n\boldsymbol{k}}\cdot\boldsymbol{\sigma}]\boldsymbol{\chi}_{n\boldsymbol{k}\pm} = \pm|\boldsymbol{S}_{n\boldsymbol{k}}|\boldsymbol{\chi}_{n\boldsymbol{k}\pm}$. It is noted that $\boldsymbol{S}_{n\boldsymbol{k}}$ is regarded as the effective spin which is felt by the n-th pair of bands. Here, we use the word "the n-th pair of bands" for the bands with indices $[n\boldsymbol{k}+]$ and $[n\boldsymbol{k}-]$. Below we shall consider the case $\mathrm{C_I}$ and $\mathrm{C_{II}}$ separately based on the above results.

Case $\mathrm{C_I}$

In this case, the orbital part of an eigenfunction is given by $\boldsymbol{u}_{n\boldsymbol{k}} = \boldsymbol{u}_n = 1/\sqrt{N_{\mathrm{sub}}}(1, e^{i\boldsymbol{b}_n\cdot\boldsymbol{a}_B}, e^{i\boldsymbol{b}_n\cdot\boldsymbol{a}_C}, \cdots)^T$, where N_{sub} is the number of sublattices, $\boldsymbol{b}_n$ is a reciprocal lattice vector, $\boldsymbol{a}_I$ is a lattice vector between I-sublattice and A-sublattice. It is noted that $\boldsymbol{u}_{n\boldsymbol{k}}$ has no $\boldsymbol{k}$-dependence, and both $\boldsymbol{S}_{n\boldsymbol{k}}$ and $\boldsymbol{\chi}_{n\boldsymbol{k}\pm}$ do not depend on the index $[n\boldsymbol{k}]$ in

this case. Using eqs. (5)-(8), σ_H is estimated as

$$\sigma_H \cong -\frac{1}{V}\sum_{\boldsymbol{k}}^{\rm 1stBZ}\sum_{n,n',n''}{}' \boldsymbol{e}_\perp\cdot\Big[\boldsymbol{\nabla}E_{n\boldsymbol{k}}\times\boldsymbol{\nabla}E_{n'\boldsymbol{k}} +\boldsymbol{\nabla}E_{n'\boldsymbol{k}}\times\boldsymbol{\nabla}E_{n''\boldsymbol{k}}+\boldsymbol{\nabla}E_{n''\boldsymbol{k}}\times\boldsymbol{\nabla}E_{n\boldsymbol{k}}\Big] \times\frac{J_H^3\Re\left[\boldsymbol{S}_{nn'}\cdot(\boldsymbol{S}_{n'n''}\times\boldsymbol{S}_{n''n})\right]}{4(E_{n\boldsymbol{k}}-E_{n'\boldsymbol{k}})(E_{n'\boldsymbol{k}}-E_{n''\boldsymbol{k}})(E_{n''\boldsymbol{k}}-E_{n\boldsymbol{k}})} \times\frac{\tau^2\left[f_F^\tau(E_{n\boldsymbol{k}})-f_F^\tau(E_{n'\boldsymbol{k}})\right]}{1+\left[(E_{n\boldsymbol{k}}-E_{n'\boldsymbol{k}})\tau\right]^2}, \tag{9}$$

where $\boldsymbol{S}_{nn'}=\sum_I u^*_{n\boldsymbol{k},I}\boldsymbol{S}_I u_{n'\boldsymbol{k},I}=\sum_I u^*_{n,I}\boldsymbol{S}_I u_{n',I}$, and the intra-pair contribution ($n=n'$) has been neglected because it is of the order of J_H^5. The symbol $\sum'_{n,n',n''}$ represents the summation with the condition that every index is different from each other. More precisely, $\boldsymbol{S}_{nn'}$ in eq. (9) is expressed as $\boldsymbol{S}_{[n\boldsymbol{k}][n'\boldsymbol{k}]}$, and $\boldsymbol{S}_{n\boldsymbol{k}}$ in eq. (6) is $\boldsymbol{S}_{[n\boldsymbol{k}][n\boldsymbol{k}]}$. However, in the case C$_{\rm I}$, $\boldsymbol{S}_{[n\boldsymbol{k}][n'\boldsymbol{k}]}$ has no $\boldsymbol{k}$-dependence.

Changing the relative scales of the parameters $|t|$, $|J_H|$, $1/\lambda_s$ and $1/\tau$, σ_H shows the crossover. Here we shall consider the following two typical cases.

C$_{\rm I}$-A: $|J_H| \ll |t|(a/\lambda_s)^d \ll 1/\tau \ll |t|(a/\lambda_s)$. This means that the length scale λ_s is shorter than the elastic-scattering length $\ell\sim|t|a\tau$, i.e. $\lambda_s\ll\ell$. In this case $|\sigma_H|\propto|J_H|^3\tau^2/|t|$.

C$_{\rm I}$-B: $1/\tau\ll|J_H|\ll|t|(a/\lambda_s)^d$, i.e., the level separation is sufficiently larger than $1/\tau$, where σ_H takes its intrinsic value.

In the former case C$_{\rm I}$-A, especially in two dimensions, we can derive the topological meaning of the AHE by relating σ_H to the skyrmion number. Because of the condition $|J_H|\ll|t|(a/\lambda_s)^d\ll1/\tau$, we can approximate the summations of band indices in eq. (9) by energy integrals, and estimated them by residues $E=\mu_0\pm i/(2\tau)$ in the complex energy-plane. (See the first formula of eq. (3).) By the inverse Fourier transformation to real-space variables, we can obtain the expression equivalent to that given by Tatara and Kawamura [20] for the periodic spin-configuration. They identified that $|J_H|\tau$ is the small parameter for the perturbative expansion of σ_H. Hence finite τ is essential there. The Hall conductivity was shown to be proportional to a sum of the spin chirality of any three localized spins with a geometrical weight in real space.

$$\sigma_H \cong \frac{e^2}{2\pi}\frac{J_H^3\tau^2}{|t|}\chi, \tag{10}$$

where χ is the total chirality in real space,

$$\chi = \frac{a^4}{V}\sum_{\boldsymbol{r}_i}\frac{\boldsymbol{e}_\perp\cdot(\boldsymbol{l}\times\boldsymbol{l}')}{ll'}I'(l)I'(l')I(l'') \times\boldsymbol{S}_{\boldsymbol{r}_1}\cdot(\boldsymbol{S}_{\boldsymbol{r}_2}\times\boldsymbol{S}_{\boldsymbol{r}_3}), \tag{11}$$

$\boldsymbol{l}=\boldsymbol{r}_1-\boldsymbol{r}_2$, $\boldsymbol{l}'=\boldsymbol{r}_2-\boldsymbol{r}_3$ and $\boldsymbol{l}''=\boldsymbol{r}_3-\boldsymbol{r}_1$. $I(r)\propto\sum_{\boldsymbol{k}}e^{i\boldsymbol{k}\cdot\boldsymbol{r}}/[1+(2\tau\xi_{\boldsymbol{k}})^2]$ is an RKKY-type function which decay exponentially in the scale of ℓ because of the complex part of the residues in energy integrals, and $I'(r)=\frac{dI(r)}{dr}$. Contribution from largely separated three spins with the scale of r decays rapidly as $\sim e^{-3r/2\ell}$, and the AHE is dominantly driven by chiralities of spins on small triangles. Using the expansion $\boldsymbol{S}_{\boldsymbol{r}_1}\cong\boldsymbol{S}_{\boldsymbol{r}_2}+\boldsymbol{l}\cdot\boldsymbol{\nabla}\boldsymbol{S}_{\boldsymbol{r}_2}$, $\boldsymbol{S}_{\boldsymbol{r}_3}\cong\boldsymbol{S}_{\boldsymbol{r}_2}-\boldsymbol{l}'\cdot\boldsymbol{\nabla}\boldsymbol{S}_{\boldsymbol{r}_2}$, we obtain

$$\chi=\frac{S^3A}{N}\int d^2\boldsymbol{r}\Phi^z(\boldsymbol{r}), \tag{12}$$

where $S=|\boldsymbol{S}_{\boldsymbol{r}}|$, N is the number of lattice sites,

$$\Phi^z(\boldsymbol{r})=\frac{1}{4\pi S^3}\boldsymbol{S}\cdot(\nabla_x\boldsymbol{S}\times\nabla_y\boldsymbol{S}) \tag{13}$$

is the skyrmion density in real space, and

$$A\sim\frac{1}{a^4}\int^{\lambda_s}dl\int^{\lambda_s}dl'l^2l'^2I'(l)I'(l')I(l''). \tag{14}$$

Because of $\lambda_s\ll\ell$, A is a dimensionless function of λ_s/a, when higher order terms in λ_s/ℓ is neglected. Thus the Hall conductivity is given by

$$\sigma_H\cong\frac{e^2}{2\pi}\cdot\frac{J_H^3\tau^2}{|t|}\cdot\frac{S^3AN_s}{N}, \tag{15}$$

where $N_s=\int d^2\boldsymbol{r}\Phi^z(\boldsymbol{r})$ is the skyrmion number [21, 22].

Case C$_{\rm II}$

Here the unit cell contains multiple sites and the periodicity of spin texture is the same as the underlying lattice. In this case, the band separation is of the order of $|t|$ except for the spin degeneracy and the degeneracy at the symmetric points. The orbital part of the eigenfunction $\boldsymbol{u}_{n\boldsymbol{k}}$ has non-trivial $\boldsymbol{k}$-dependence, and the same goes for $S_{n\boldsymbol{k}}$ and $\boldsymbol{\chi}_{n\boldsymbol{k}}$. As we shall see below, the intra-pair contribution, which contains matrix element $\langle\boldsymbol{\psi}_{n\boldsymbol{k}\mp}|\boldsymbol{J}|\boldsymbol{\psi}_{n\boldsymbol{k}\pm}\rangle$, is of the order of $|J_H|^3\tau^2/|t|$ or $|J_H|/|t|$ depending on $|J_H|\tau\ll1$ or $\gg1$. The inter-pair contribution, which contains the matrix element $\langle\boldsymbol{\psi}_{n\boldsymbol{k}s}|\boldsymbol{J}|\boldsymbol{\psi}_{n'\boldsymbol{k}s'}\rangle$ ($n\neq n'$), is of the order of $|J_H|^3/|t|^3$, and thus negligibly small compared with the intra-pair contribution. Therefore, the dominant contributions to σ_H come from the intra-pair terms in sharp contrast to the case C$_{\rm I}$.

Considering that the n-th pair of bands is intersecting the Fermi level, the dominant contribution is given by

$$\sigma_H \cong -\frac{2i}{V}\sum_{\boldsymbol{k}}^{\rm 1stBZ}\boldsymbol{e}_\perp\cdot[\langle\boldsymbol{\psi}_{n\boldsymbol{k}+}|\boldsymbol{J}|\boldsymbol{\psi}_{n\boldsymbol{k}-}\rangle\times\langle\boldsymbol{\psi}_{n\boldsymbol{k}-}|\boldsymbol{J}|\boldsymbol{\psi}_{n\boldsymbol{k}+}\rangle] \times\frac{\tau^2\left[f_F^\tau(E_{n\boldsymbol{k}+})-f_F^\tau(E_{n\boldsymbol{k}-})\right]}{1+\left[(E_{n\boldsymbol{k}-}-E_{n\boldsymbol{k}+})\tau\right]^2}. \tag{16}$$

The perturbative eigenfunctions give the following matrix element of the current operator

$$\langle\boldsymbol{\psi}_{n\boldsymbol{k}\mp}|\boldsymbol{J}|\boldsymbol{\psi}_{n\boldsymbol{k}\pm}\rangle\cong-\frac{e}{2}J_H|\boldsymbol{S}_{n\boldsymbol{k}}|\boldsymbol{\chi}^\dagger_{n\boldsymbol{k}\mp}\left[\boldsymbol{\nabla}_{\boldsymbol{k}}\boldsymbol{e}_{n\boldsymbol{k}}\cdot\boldsymbol{\sigma}\right]\boldsymbol{\chi}_{n\boldsymbol{k}\pm}, \tag{17}$$

where the higher order terms in J_H/t are neglected, and $\boldsymbol{e}_{n\boldsymbol{k}} = \boldsymbol{S}_{n\boldsymbol{k}}/|\boldsymbol{S}_{n\boldsymbol{k}}|$.

Using the above result and the completeness condition,

$$\sum_{s=+,-} \boldsymbol{\chi}_{n\boldsymbol{k}s} \otimes \boldsymbol{\chi}^{\dagger}_{n\boldsymbol{k}s} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{18}$$

we can estimate σ_H as follows,

$$\sigma_H \cong \frac{e^2}{2V} \sum_{\boldsymbol{k}}^{\text{1stBZ}} \left[-\frac{df_F^{\tau}}{dE}(E_{n\boldsymbol{k}}) \right] \boldsymbol{e}_{\perp} \cdot \boldsymbol{\Phi}_{n\boldsymbol{k}} \times \frac{(J_H|\boldsymbol{S}_{n\boldsymbol{k}}|)^3\tau^2}{1+(J_H|\boldsymbol{S}_{n\boldsymbol{k}}|\tau)^2}, \tag{19}$$

where

$$\Phi^{\lambda}_{n\boldsymbol{k}} = \frac{1}{2}\sum_{\mu,\nu} \epsilon^{\lambda\mu\nu} \boldsymbol{e}_{n\boldsymbol{k}} \cdot \left(\nabla_{k_\mu} \boldsymbol{e}_{n\boldsymbol{k}} \times \nabla_{k_\nu} \boldsymbol{e}_{n\boldsymbol{k}} \right). \tag{20}$$

It is noted that $\boldsymbol{\Phi}_{n\boldsymbol{k}}$ may be regarded as the effective chirality of the n-th pair of bands in momentum space, because its integration is related to the solid angle of $\boldsymbol{S}_{n\boldsymbol{k}}$. Especially in a two-dimensional system, $\boldsymbol{e}_{\perp} \cdot \boldsymbol{\Phi}_{n\boldsymbol{k}}/(4\pi)$ represent the skyrmion density in momentum space. This means that σ_H is characterized by the momentum-space skyrmion-density at the Fermi level. When we change the parameter $|J_H|\tau$, σ_H shows the crossover,

$$|\sigma_H| \propto \begin{cases} \frac{|J_H|^3\tau^2}{|t|}, & (|J_H|\tau \ll 1) \\ \frac{|J_H|}{|t|}, & (|J_H|\tau \gg 1) \end{cases}. \tag{21}$$

Finally, it is noted that, in contrast to the case $\mathrm{C_I}$, the AHE takes place even if there is no winding of spin texture in real space and the momentum-space skyrmion-number is also zero.

In order to confirm the above consideration for the case $\mathrm{C_{II}}$, we present the explicit results in Fig. 1 for the model eq. (1) on the kagome lattice with the spin texture,

$$\frac{\boldsymbol{S}_I}{S} = (\sin\theta\cos\phi_I, \sin\theta\sin\phi_I, \cos\theta), \tag{22}$$

where $\phi_I = (4n_I - 1)\pi/6$ ($n_A = 1, n_B = 2, n_C = 3$). The results are calculated by using the first line of eq. (3), and clearly shows the crossover as predicted in eq. (21).

In conclusion, we have shown that the non-coplanar spin configuration induces the AHE by two distinct mechanisms. The AHE is characterized by the real-space skyrmion-number when the underlying lattice structure is irrelevant and the spin texture is slowly varying as $a \ll \lambda_s \ll \ell$. On the other hand, the AHE is characterized by the momentum-space skyrmion-density at the Fermi level when the underlying lattice structure is relevant and the periodicity of the spin texture is the same as the lattice. The Hall resistivity and conductivity become essential, i.e., independent of the elastic-scattering time, for $|J_H|\tau \ll 1$ and for $|J_H|\tau \gg 1$ respectively.

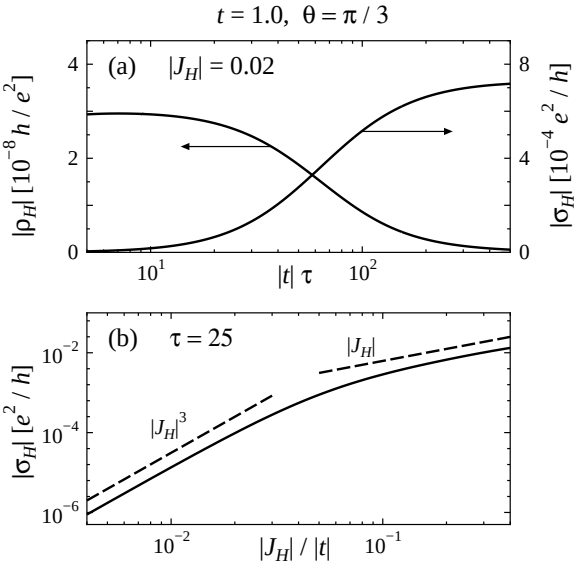

FIG. 1: (a) τ-dependence of ρ_H and σ_H, and (b) $|J_H|$-dependence of σ_H for the kagome-lattice model in the case where the average number of electrons per unit cell is three. It is shown in the panel (a) that ρ_H and σ_H approach to constant values for $|J_H|\tau \ll 1$ and for $|J_H|\tau \gg 1$ respectively. The panel (b) shows the crossover from $|\sigma_H| \propto |J_H|^3$ for $|J_H|\tau \ll 1$ to $|\sigma_H| \propto |J_H|$ for $|J_H|\tau \gg 1$.

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